

On γ -Regular-Open Sets and γ -Closed Spaces

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Abstract. The purpose of this paper is to continue studying the properties of γ -regular open sets introduced and explored in [6]. The concept of γ -closed spaces have also been defined and discussed.

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1 Introduction

The concept of operation γ was initiated by S. Kasahara [7]. He also introduced γ -closed graph of a function. Using this operation, H. Ogata [8] introduced the concept of γ -open sets and investigated the related topological properties of the associated topology τ_γ and τ . He further investigated general operator approaches of close graph of mappings.

Further S. Hussain and B. Ahmad [1-6] continued studying the properties of γ -open(closed) sets and generalized many classical notions in their work. The purpose of this paper is to continue studying the properties of γ -regular open sets introduced and explored in [6]. The concept of γ -closed spaces have also been defined and discussed.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space

in place of a topological space.

2 Preliminaries

Throughout the present paper, X denotes topological spaces.

Definition [7]. An operation $\gamma : \tau \rightarrow P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V . The operations defined by $\gamma(G) = G$, $\gamma(G) = \text{cl}(G)$ and $\gamma(G) = \text{intcl}(G)$ are examples of operation γ .

Definition [7]. Let $A \subseteq X$. A point $x \in A$ is said to be γ -interior point of A , if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $\text{int}_\gamma(A)$. Thus

$$\text{int}_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ -open [8] iff $A = \text{int}_\gamma(A)$. A set A is called γ -closed [1] iff $X-A$ is γ -open.

Definition [1]. A point $x \in X$ is called a γ -closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \emptyset$, for each open nbd U of x . The set of all γ -closure points of A is called γ -closure of A and is denoted by $\text{cl}_\gamma(A)$. A subset A of X is called γ -closed, if $\text{cl}_\gamma(A) \subseteq A$. Note that $\text{cl}_\gamma(A)$ is contained in every γ -closed superset of A .

Definition [7]. An operation γ on τ is said be regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition [8]. An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists γ -open set B such that $x \in B$ and $U^\gamma \supseteq B$.

3 γ -Regular-Open Sets

Definition 3.1 [6]. A subset A of X is said to be γ -regular-open (respt. γ -regular-closed), if $A = \text{int}_\gamma(\text{cl}_\gamma(A))$ (respt. $A = \text{cl}_\gamma(\text{int}_\gamma(A))$).

It is clear that $RO_\gamma(X, \tau) \subseteq \tau_\gamma \subseteq \tau$ [6].

The following example shows that the converse of above inclusion is not true in general.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \rightarrow P(X)$ by

$$\gamma(A) = \begin{cases} A, & \text{if } b \in A \\ cl_\gamma(A), & \text{if } b \notin A \end{cases}$$

Calculations shows that $\{a, b\}, \{a, c\}, \{b\}, X, \phi$ are γ -open sets and $\{a, c\}, \{b\}, X, \phi$ are γ -regular-open sets. Here set $\{a, b\}$ is γ -open but not γ -regular-open.

Definition 3.3[7]. A space X is called γ -extremally disconnected, if for all γ -open subset U of X , $cl_\gamma(U)$ is a γ -open subset of X .

Proposition 3.4. If A is a γ -clopen set in X , then A is a γ -regular-open set. Moreover, if X is γ -extremally disconnected then the converse holds.

Proof. If A is a γ -clopen set, then $A = cl_\gamma(A)$ and $A = int_\gamma(A)$, and so we have $A = int_\gamma(cl_\gamma(A))$. Hence A is γ -regular-open.

Suppose that X is a γ -extremally disconnected space and A is a γ -regular-open set in X . Then A is γ -open and so $cl_\gamma(A)$ is a γ -open set. Hence $A = int_\gamma(cl_\gamma(A)) = cl_\gamma(A)$ and hence A is γ -closed set. This completes the proof.

The following example shows that space X to be γ -extremally disconnected is necessary in the converse of above Proposition.

Example 3.5 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Define an operation $\gamma : \tau \rightarrow P(X)$ by $\gamma(B) = int(cl(B))$. Clearly X is not γ -extremally disconnected space. Calculations shows that $\{a\}, \{a, b\}, \{b\}, X, \phi$ are γ -open as well as γ -regular-open sets. Here $\{a\}$ is a γ -regular-open set but not γ -clopen set.

Theorem 3.6. Let $A \subseteq X$, then $(a) \Rightarrow (b) \Rightarrow (c)$, where :

- (a) A is γ -clopen.
- (b) $A = cl_\gamma(int_\gamma(A))$.
- (c) $X - A$ is γ -regular-open.

Proof. $(a) \Rightarrow (b)$. This is obvious.

$(b) \Rightarrow (c)$. Let $A = cl_\gamma(int_\gamma(A))$. Then $X - A = X - cl_\gamma(int_\gamma(A)) = int_\gamma(X - int_\gamma(A)) = int_\gamma(cl_\gamma(X - A))$, and hence $X - A$ is γ -regular-open set. Hence the proof.

Using Proposition 3.4, we have the following Theorem:

Theorem 3.7. If X is a γ -extremally disconnected space. Then $(a) \Rightarrow (b) \Rightarrow (c)$, where :

- (a) $X - A$ is γ -regular-open.
- (b) A is γ -regular-open.
- (c) A is γ -clopan.

Proof. $(a) \Rightarrow (b)$. Suppose X is γ -extremally disconnected space. From Proposition 3.4, $X - A$ is a γ -open and γ -closed set, and hence A is a γ -open and γ -closed set. Thus $A = \text{int}_\gamma(\text{cl}_\gamma(A))$ implies A is γ -regular-open set.

$(b) \Rightarrow (c)$. This directory follows from Proposition 3.4. This completes as required.

Combining Theorems 3.6 and 3.7, we have the following:

Theorem 3.8. If X is a γ -extremally disconnected space. Then the following statements are equivalent:

- (a) A is γ -clopan.
- (b) $A = \text{cl}_\gamma(\text{int}_\gamma(A))$.
- (c) $X - A$ is γ -regular-open.
- (d) A is γ -regular-open.

Theorem 3.9. Let $A \subseteq X$ and γ be an open operation. If $\text{cl}_\gamma(A)$ is a γ -regular-open set. Then A is a γ -open set in X . Moreover, if X is extremally γ -disconnected then the converse holds.

Proof. Suppose that $\text{cl}_\gamma(A)$ is a γ -regular-open sets. Since γ is open, we have

$A \subseteq \text{cl}_\gamma(A) \subseteq \text{int}_\gamma(\text{cl}_\gamma(\text{cl}_\gamma(A))) = \text{int}_\gamma(\text{cl}_\gamma(A)) = \text{int}_\gamma(A)$. This implies that A is γ -open set.

Suppose that X is γ -extremally disconnected and A is γ -open set. Then $\text{cl}_\gamma(A)$ is a γ -open set, and hence γ -clopan set. Thus by Theorem 3.8, $\text{cl}_\gamma(A)$ is a γ -regular-open set. This completes the proof.

Corollary 3.10. Let X be a γ -extremally disconnected space. Then for each subset A of X , the set $\text{cl}_\gamma(\text{int}_\gamma(A))$ is γ -regular-open sets.

Definition 3.11. A point $x \in X$ is said to be a γ - θ -cluster point of a subset A of X , if $cl_\gamma(U) \cap A \neq \emptyset$ for every γ -open set U containing x . The set of all γ - θ -cluster points of A is called the γ - θ -closure of A and is denoted by $\gamma cl_\theta(A)$.

Definition 3.12. A subset A of X is said to be γ - θ -closed, if $\gamma cl_\theta(A) = A$. The complement of γ - θ -closed set is called γ - θ -open sets. Clearly a γ - θ -closed (γ - θ -open) is γ -closed(γ -open) set.

Proposition 3.13. Let A and B be subsets of a space X . Then the following properties hold:

- (1) If $A \subseteq B$, then $\gamma cl_\theta(A) \subseteq \gamma cl_\theta(B)$.
- (2) If A_i is γ - θ -closed in X , for each $i \in I$, then $\bigcap_{i \in I} A_i$ is γ - θ -closed in X .

Proof. (1). This is obvious.

(2). Let A_i be a γ - θ -closed in X for each $i \in I$. Then $A_i = \gamma cl_\theta(A_i)$ for each $i \in I$. Thus we have $\gamma cl_\theta(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} \gamma cl_\theta(A_i) = \bigcap_{i \in I} A_i \subseteq \gamma cl_\theta(\bigcap_{i \in I} A_i)$.

Therefore, we have $\gamma cl_\theta(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} A_i$ and hence $\bigcap_{i \in I} A_i$ is γ - θ -closed. Hence the proof.

Theorem 3.14. If γ is an open operation. Then for any subset A of γ -extremally disconnected space X , the following hold:

$$\begin{aligned} \gamma cl_\theta(A) &= \bigcap \{V : A \subseteq V \text{ and } V \text{ is } \gamma\text{-}\theta\text{-closed}\} \\ &= \bigcap \{V : A \subseteq V \text{ and } V \text{ is } \gamma\text{-regular-open}\} \end{aligned}$$

Proof. Let $x \notin \gamma cl_\theta(A)$. Then there is a γ -open set V with $x \in V$ such that $cl_\gamma(V) \cap A = \emptyset$. By Theorem 3.9, $X - cl_\gamma(V)$ is γ -regular-open and hence $X - cl_\gamma(V)$ is a γ - θ -closed set containing A and $x \notin X - \gamma cl_\theta(V)$. Thus we have $x \notin \bigcap \{V : A \subseteq V \text{ and } V \text{ is } \gamma\text{-}\theta\text{-closed}\}$.

Conversely, suppose that $x \notin \bigcap \{V : A \subseteq V \text{ and } V \text{ is } \gamma\text{-}\theta\text{-closed}\}$. Then there exists a γ - θ -closed set V such that $A \subseteq V$ and $x \notin V$, and so there exists a γ -open set U with $x \in U$ such that $U \subseteq cl_\gamma(U) \subseteq X - V$. Thus we have $cl_\gamma(U) \cap A \subseteq cl_\gamma(U) \cap V = \emptyset$ implies $x \notin \gamma cl_\theta(A)$.

The proof of the second equation follows similarly. This completes the proof.

Theorem 3.15. Let γ be an open operation. If X is a γ -extremally disconnected space and $A \subseteq X$. Then the followings hold:

- (a) $x \in \gamma cl_\theta(A)$ if and only if $V \cap A \neq \emptyset$, for each γ -regular-open set V with $x \in V$.

(b) A is γ - θ -open if and only if for each $x \in A$ there exists a γ -regular-open set V with $x \in V$ such that $V \subseteq A$.

(c) A is a γ -regular-open set if and only if A is γ - θ -clopan.

Proof. (a) and (b) follows directly from Theorems 3.8 and 3.9.

(c) Let A be a γ -regular-open set. Then A is a γ -open set and so $A = cl_\gamma(A) = \gamma cl_\theta(A)$ and hence A is γ - θ -closed. Since $X - A$ is a γ -regular-open set, by the argument above, $X - A$ is γ - θ -closed and A is γ - θ -open. The converse is obvious. Hence the proof.

It is obvious that γ -regular-open $\Rightarrow \gamma$ - θ -open $\Rightarrow \gamma$ -open. But the converses are not necessarily true as the following examples show.

Example 3.16. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \rightarrow P(X)$ by

$$\gamma(A) = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A \end{cases}$$

Calculations shows that $\{a, b\}, \{a, c\}, \{b\}, X, \phi$ are γ -open sets as well as γ - θ -open sets and γ -regular-open sets are $\{a, c\}, \{b\}, X, \phi$. Then the subset $\{a, b\}$ is γ - θ -open but not γ -regular-open.

Example 3.17. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be a topology on X . For $b \in X$, define an operation $\gamma : \tau \rightarrow P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } b \in A \\ A, & \text{if } b \notin A \end{cases}$$

Calculations shows that $\{\phi, X, \{a\}, \{a, c\}\}$ are γ -open sets and $\{\phi, X, \{a, c\}\}$ are γ - θ -open sets. The subset $\{a\}$ is γ -open but not γ - θ -open.

4 γ -Closed Spaces

Definition 4.1. A filterbase Γ in X , γ -R-converges to $x_0 \in X$, if for each γ -regular-open set A with $x_0 \in A$, there exists $F \in \Gamma$ such that $F \subseteq A$.

Definition 4.2. A filterbase Γ in X γ -R-accumulates to $x_0 \in X$, if for each γ -regular-open set A

with $x_0 \in A$ and each $F \in \Gamma$, $F \cap A \neq \phi$.

The following Theorems directly follow from the above definitions.

Theorem 4.3. If a filterbase Γ in X , γ -R-converges to $x_0 \in X$, then Γ γ -R-accumulates to x_0 .

Theorem 4.4. If Γ_1 and Γ_2 are filterbases in X such that Γ_2 subordinate to Γ_1 and Γ_2 γ -R-accumulates to x_0 , then Γ_1 γ -R-accumulates to x_0 .

Theorem 4.5. If Γ is a maximal filterbase in X , then Γ γ -R-accumulates to x_0 if and only if Γ γ -R-converges to x_0 .

Definition 4.6. A space X is said to be γ -closed, if every cover $\{V_\alpha : \alpha \in I\}$ of X by γ -open sets has a finite subset I_0 of I such that $X = \bigcup_{\alpha \in I_0} cl_\gamma(V_\alpha)$.

Proposition 4.7. If γ is an open operation, Then the following are equivalent:

- (1) X is γ -closed.
- (2) For each family $\{A_\alpha : \alpha \in I\}$ of γ -closed subsets of X such that $\bigcap_{\alpha \in I} A_\alpha = \phi$, there exists a finite subset I_0 of I such that $\bigcap_{\alpha \in I_0} int_\gamma(A_\alpha) = \phi$.
- (3) For each family $\{A_\alpha : \alpha \in I\}$ of γ -closed subsets of X , if $\bigcap_{\alpha \in I_0} int_\gamma(A_\alpha) \neq \phi$, for every finite subset I_0 of I , then $\bigcap_{\alpha \in I} A_\alpha \neq \phi$.
- (4) Every filterbase Γ in X γ -R-accumulates to $x_0 \in X$.
- (5) Every maximal filterbase Γ in X γ -R-converges to $x_0 \in X$.

Proof. (2) \Leftrightarrow (3). This is obvious.

(2) \Rightarrow (1). Let $\{A_\alpha : \alpha \in I\}$ be a family of γ -open subsets of X such that $X = \bigcup_{\alpha \in I} A_\alpha$. Then each $X - A_\alpha$ is a γ -closed subset of X and $\bigcap_{\alpha \in I} (X - A_\alpha) = \phi$, and so there exists a finite subset I_0 of I such that $\bigcap_{\alpha \in I_0} int_\gamma(X - A_\alpha) = \phi$, and hence $X = \bigcup_{\alpha \in I_0} (X - int_\gamma(X - A_\alpha)) = \bigcup_{\alpha \in I_0} cl_\gamma(A_\alpha)$. Therefore X is γ -closed, since γ is open.

(4) \Rightarrow (2). Let $\{A_\alpha : \alpha \in I\}$ be a family of γ -closed subsets of X such that $\bigcap_{\alpha \in I} A_\alpha = \phi$. Suppose that for every finite subfamily $\{A_{\alpha_i} : i = 1, 2, \dots, n\}$, $\bigcap_{i=1}^n int_\gamma(A_{\alpha_i}) \neq \phi$. Then $\bigcap_{i=1}^n (A_{\alpha_i}) \neq \phi$ and $\Gamma = \{\bigcap_{i=1}^n A_{\alpha_i} : n \in N, \alpha_i \in I\}$ forms a filterbase in X . By (4), Γ γ -R-

accumulates to some $x_0 \in X$. Thus for every γ -open set A with $x_0 \in A$ and every $F \in \Gamma$, $F \cap cl_\gamma(A) \neq \phi$. Since $\bigcap_{F \in \Gamma} F = \phi$, there exists a $F \in \Gamma$ such that $x_0 \notin F$, and so there exists $\alpha_0 \in I$ such that $x_0 \notin A_{\alpha_0}$ and hence $x_0 \in X - A_{\alpha_0}$ and $X - A_{\alpha_0}$ is a γ -open set. Thus $x_0 \notin int_\gamma(A_{\alpha_0})$ and $x_0 \in X - int_\gamma(A_{\alpha_0})$, and hence $F_0 \cap (X - int_\gamma(A_{\alpha_0})) = F_0 \cap cl_\gamma(X - A_{\alpha_0}) = \phi$, which is a contradiction to our hypothesis.

(5) \Rightarrow (4). Let Γ be filterbase in X . Then there exists a maximal filterbase ξ in X such that ξ subordinate to Γ . Since ξ γ -R-converges to x_0 , so by Theorems 4.4 and 4.5, Γ γ -R-accumulate to x_0 .

(1) \Rightarrow (5). Suppose that $\Gamma = \{F_a : a \in I\}$ is a maximal filterbase in X which does not γ -R-converge to any point in X . From Theorem 4.5, Γ does not γ -R-accumulates at any point in X . Thus for every $x \in X$, there exists a γ -open set A_x containing x and $F_{a_x} \in \Gamma$ such that $F_{a_x} \cap cl_\gamma(A_x) = \phi$. Since $\{A_x : x \in X\}$ is γ -open cover of X , there exists a finite subfamily $\{A_{x_i} : i = 1, 2, \dots, n\}$ such that $X = \bigcup_{i=1}^n cl_\gamma(A_{x_i})$. Because Γ is a filterbase in X , there exists $F_0 \in \Gamma$ such that $F_0 \subseteq \bigcap_{i=1}^n F_{a_{x_i}}$, and hence $F_0 \cap cl_\gamma(A_{x_i}) = \phi$ for all $i = 1, 2, \dots, n$. Hence we have that, $\phi = F_0 \cap (\bigcup_{i=1}^n cl_\gamma(A_{x_i})) = F_0 \cap X$, and hence $F_0 = \phi$. This is a contradiction. Hence the proof.

Definition 4.8. A net $(x_i)_{i \in D}$ in a space X is said to be γ -R-converges to $x \in X$, if for each γ -open set U with $x \in U$, there exists i_0 such that $x_i \in cl_\gamma(U)$ for all $i \geq i_0$, where D is a directed set.

Definition 4.9. A net $(x_i)_{i \in D}$ in a space X is said to be γ -R-accumulates to $x \in X$, if for each γ -open set U with $x \in U$ and each i , $x_i \in cl_\gamma(U)$, where D is a directed set.

The proofs of following Propositions are easy and thus are omitted:

Proposition 4.10. Let $(x_i)_{i \in D}$ be a net in X . For the filterbase $F((x_i)_{i \in D}) = \{\{x_i : i \leq j\} : j \in D\}$ in X ,

- (1) $F((x_i)_{i \in D})$ γ -R-converges to x if and only if $(x_i)_{i \in D}$ γ -R-converges to x .
- (2) $F((x_i)_{i \in D})$ γ -R-accumulates to x if and only if $(x_i)_{i \in D}$ γ -R-accumulates to x .

Proposition 4.11. Every filterbase F in X determines a net $(x_i)_{i \in D}$ in X such that

- (1) F γ -R-converges to x if and only if $(x_i)_{i \in D}$ γ -R-converges to x .
- (2) F γ -R-accumulates to x if and only if $(x_i)_{i \in D}$ γ -R-accumulates to x .

From Propositions 4.11 and 4.12, filterbases and nets are equivalent in the sense of γ -R-converges and γ -R-accumulates. Thus we have the following Theorem:

Theorem 4.13. For a space X , the following are equivalent:

- (1) X is γ -closed.
- (2) Each net $(x_i)_{i \in D}$ in X has a γ -R-accumulation point.
- (3) Each universal net in X γ -R-converges.

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